The calculation of eclipses is chiefly a combination of geometry and orbital mechanics. Hence, it is necessary to understand the orbital mechanics involved and also the underlying geometry.

Orbital Mechanics

You are encouraged to look into an undergraduate or graduate level textbook for details. Consider the orbital mechanics of the motion of two masses M_1 and M_2. It can be shown that this two body problem can be reduced to a single body problem of the reduced mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$ going around the total mass of the system $(M_1 + M_2)$ located at the center of mass. The orbit of the reduced mass is an ellipse that is described two parameters: the semi-major axis a and the eccentricity e. For clarity, a picture of an ellipse is shown in Fig 1. The semi-major axis is given by the length OA. The center of mass is located at one of the foci of the ellipse, shown as F. The equation of the ellipse is given by

$$r = \frac{a(1 - e^2)}{1 + e \cos\theta} \tag{1}$$

where r and θ are as shown in the figure.

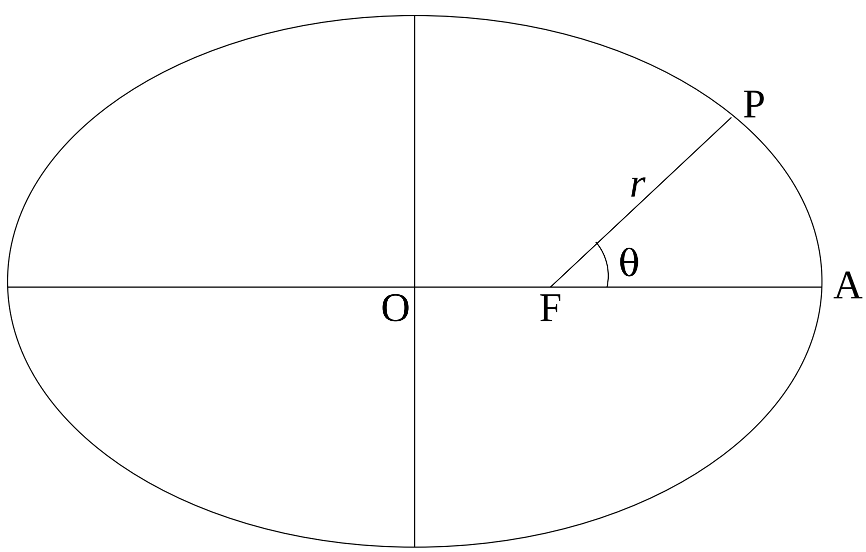

Figure 1: Schematic of an ellipse

In order to simulate the orbit of the masses, you will also need to know the angular momentum of the system. The angular momentum L is a conserved quantity and is given by

$$L = \mu r^2 \frac{d\theta}{dt} \tag{2}$$

In addition, it can be shown that the angular momentum in terms of orbital parameters is given by

$$L = \mu\sqrt{GMa(1-e^2)} \tag{3}$$

Using this information, you can compute the orbit of the reduced mass as a function of time as follows: Compute the angular momentum of the system using equation (3). Now, from equation (2), $d\theta = \frac{L}{\mu r^2}dt$. Hence, starting from an initial condition, you can compute the angle θ at any instant using a time step dt. Then, use equation (1) to compute r, which fully characterizes the location of the reduced mass.

The velocity of the reduced mass at a given r is given by

$$v^2 = G(M_1 + M_2)\left(\frac{2}{r} - \frac{1}{a}\right) \tag{4}$$

The position vectors and velocities of the two masses can be got from the corresponding quantities of the reduced mass by the following equations:

$$\vec{r_1} = -\frac{\mu}{M_1}\,\vec{r} \tag{5}$$

$$\vec{v_1} = -\frac{\mu}{M_1}\,\vec{v} \tag{6}$$

$$\vec{r_2} = \frac{\mu}{M_2}\,\vec{r} \tag{7}$$

$$\vec{v_2} = \frac{\mu}{M_2}\,\vec{v} \tag{8}$$

Note: In the case of orbit of planets around the Sun, the mass of the Sun is much greater than the mass of the planets. Hence, in the case of the Sun ($\odot$) – Earth ($\oplus$) system, assuming $M_1 = M_\odot$ and $M_2 = M_\oplus$, since $M_\odot \gg M_\oplus$, one can assume that $\mu = M_\oplus$, and $M_1 + M_2 = M_\odot$. Hence the assumption that the Sun is at rest at the origin ($\vec{v_1} = 0$, $\vec{r_1} = 0$). This should simplify the calculations of the Sun-Earth orbit a little bit.

Geometry of Moon's orbit with respect to Earth's orbit

The plane of the orbit of Earth around the Sun is called the *ecliptic*. The orbit of the Moon around the Earth is tilted to the ecliptic by 5°. In the same way as the tilt of the Earth's axis remains fixed with respect to the stars as the Earth goes the Sun, the orientation of the plane of Moon's orbit remains the same with respect to the stars as the Earth goes around the Sun. The various planes are shown in figure 2 for clarity.

In order to do the eclipse calculations, one should be able to determine the positions of the Earth and Moon in three dimensions at any instant of time. The positions of Earth can be done easily using its orbital parameters. For the Moon, this has to be done in two steps: first the position of the Moon with the Earth as the origin can be calculated in the same way as the calculation of position of Earth around the Sun (one has to put in the

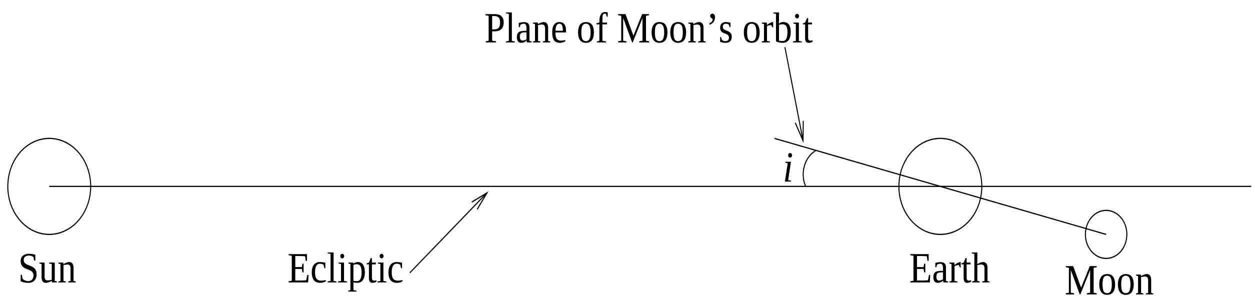

Figure 2: Geometry of various orbits. The sizes and inclinations have been exaggerated for clarity.

orbital parameters of Moon). Then, a coordinate transformation has to be done to tilt the system by the inclination i (which adds the third dimension) and to shift the origin to the Sun.

Orbital parameters

$M_\odot = 1.989 \times 10^{33}$ g
$M_\oplus = 5.974 \times 10^{27}$ g
$M_{moon} = 7.35 \times 10^{25}$ g

Orbital parameters of Earth-Sun system:
$a = 1.496 \times 10^{13}$ cm
$P = 365.256$ days
$e = 0.0167$

Orbital parameters of Earth-Moon system:
$a = 3.844 \times 10^{10}$ cm
$P = 27.3217$ days
$i = 5.145°$
$e = 0.0549$

The data above is taken from http://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html and http://nssdc.gsfc.nasa.gov/planetary/factsheet/moonfact.html.

Eclipse Geometry

The geometry of the eclipse is shown in figure 3. The object 1 represents the Sun and the object 2 represents the Moon in the case of solar eclipse or the Earth in the case of a lunar eclipse. The cone that converges after object 2 is called the umbra, and the opening angle of this cone is pointed as θ_u. The cone that diverges with an apex between the two objects is called the penumbra and has a half angle shown as θ_p. The two angles are approximately related to the radii of the two objects as

$$\tan\theta_u = \frac{R_1 - R_2}{d} \quad (9)$$

$$\tan\theta_p = \frac{R_1 + R_2}{d} \quad (10)$$

where d is the distance between the centers of the two objects.

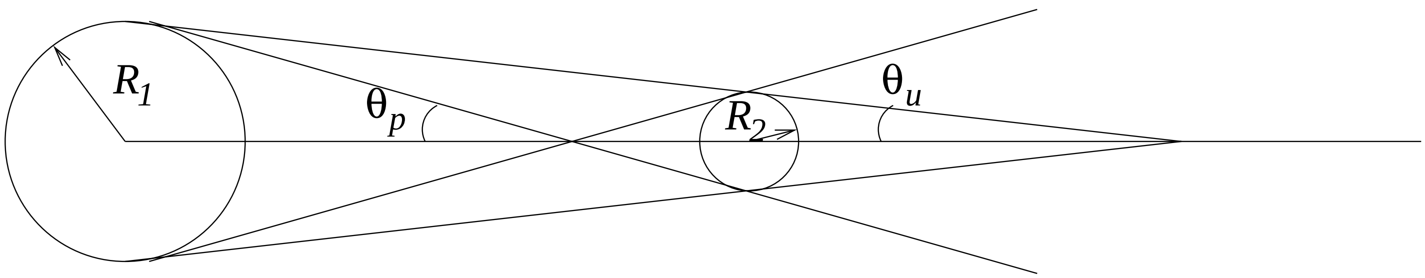

Figure 3: Eclipse geometry showing umbra and penumbra

The challenge in eclipse calculation is to determine whether Earth will lie within the umbra or penumbra of the Moon (vice versa for lunar eclipse) using the three dimensional positions of the Earth and the Moon. It is quite non-trivial and is a chief ingredient in the eclipse code. The radii of the Sun, Earth and Moon are given below:

$R_\odot = 6.9599 \times 10^{10}$ cm
$R_\oplus = 6.378 \times 10^8$ cm
$R_{moon} = 1.738 \times 10^8$ cm

I assume that you are not trying to do an eclipse calculation exactly. If so, the problem becomes *much* more complicated, as you have to take into account the perturbations of all planets, the exact relation between θ_p, θ_u and the radii R_1, R_2, etc. In addition, you will need the initial condition of the Earth-Moon-Sun system, which you can obtain over the internet. However, to get a rough idea as to modeling the eclipse calculations, you can possibly approximate the orbits a circles, as the eccentricities are very small. Good luck on your software development effort!

Regards,
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